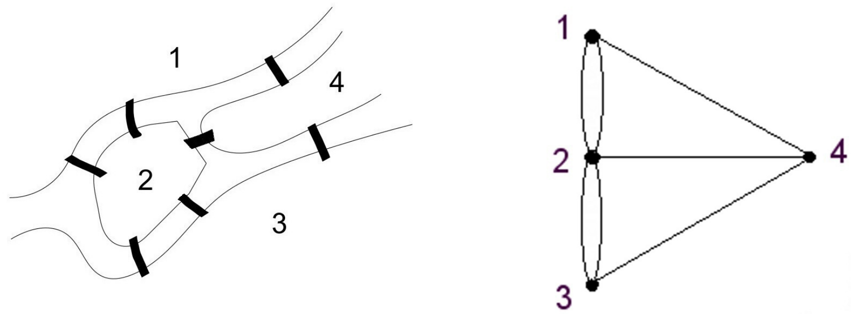

Figure 1: Euler's problem of the bridges and a suitable graph

1 Principles of Graph Theory

With his problem of the bridges of Königsberg, Leonhard Euler (1707 - 1783) established the mathematical discipline of graph theory. The elements of a graph are vertices and edges. The graph on the right in figure 1 shows four vertices which represent the four regions of mainland and seven edges which represent the seven bridges over the river Pregel.

Definition A.1 A *directed graph* or *digraph* $\mathbf{G}$ is an ordered pair $(\mathbf{V}, \mathbf{E})$, where $\mathbf{V}$ is a (finite or denumerably infinite) set of vertices and $\mathbf{E}$ is a set of edges.
The $incidence function$: $\mathbf{E} \to (\mathbf{V} \times \mathbf{V})$ assigns every edge $\mathbf{e} \in \mathbf{E}$ an ordered pair $(\mathbf{u}, \mathbf{v}) \in (\mathbf{V} \times \mathbf{V})$. The pair $\mathbf{e} \equiv (\mathbf{u}, \mathbf{v})$ is called a *directed edge*. We say $\mathbf{e}$ and $\mathbf{u}$ are *incident* and $\mathbf{e}$ and $\mathbf{v}$ are *incident*. We call $\mathbf{G}$ finite, if $\mathbf{V}$ and $\mathbf{E}$ are finite sets.
The two *endpoints* of an edge $(\mathbf{u}, \mathbf{v})$ are called *initial endpoint* (tail) $\mathbf{p}_{\text{init}} := \mathbf{u}$ and *terminal endpoint* (head) $\mathbf{p}_{\text{term}} := \mathbf{v}$. In a diagram an arrowhead is placed at $\mathbf{p}_{\text{term}}$. Two edges are called *adjacent* if they have an endpoint in common. Two vertices are called *adjacent* ($\sim$), if there is an edge which is incident to both. An edge with identical endpoints is called *loop*.

Definition A.2 If every edge of a graph $\mathbf{G}$ is undirected, i. e. of the form $\{\mathbf{u}, \mathbf{v}\}$, then $\mathbf{G}$ is called an *undirected graph*.

Definition A.3 A graph is called *simple*, if for any $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ we have

(S1) at most one of the pairs $\mathbf{e} := (\mathbf{u}, \mathbf{v})$, $\bar{\mathbf{e}} := (\mathbf{v}, \mathbf{u})$ or $\mathbf{f} := \{\mathbf{v}, \mathbf{u}\}$ is an element of $\mathbf{E}$ (no *multiple edges*) and

(S2) a pair of the form $(\mathbf{v},\ \mathbf{v})$ or $\{\mathbf{v}, \mathbf{v}\}$ is not an element of $\mathbf{E}$ (no *loops*).

Definition A.4 An *oriented graph* is a simple directed graph.

Definition A.5 A simple undirected graph is an oriented graph, the orientations of whose edges are abolished, i. e. every edge is a set of the form $\{\mathbf{u}, \mathbf{v}\}$.

On the other hand we get an oriented graph by assigning every edge of an unoriented simple graph a direction. However this is ambiguous. It can be done in $2^{|\mathrm{E}|}$ different ways. Thus there are $2^{|\mathrm{E}|}$ different oriented graphs over the same simple unoriented graph.

Degrees of a vertex A.6 The *degree* $\deg(\mathbf{v})$ of a vertex $\mathbf{v}$ is the number of edges that are incident to $\mathbf{v}$. Loops are counted twice. A vertex of degree 0 is called a *lone vertex*. For every vertex $\mathbf{v}$ of a directed graph we distinguish between its *exit degree* $\deg^+(\mathbf{u}) = |\{\mathbf{v}|(\mathbf{u}, \mathbf{v}) \in \mathbf{E}\}|$ and its *initial degree* $\deg^-(\mathbf{u}) = |\{\mathbf{v}|(\mathbf{v}, \mathbf{u}) \in \mathbf{E}\}|$.

We have that $\mathrm{d}^+(\mathbf{u})$ equals the number of initial vertices in a given graph. Since every directed edge has exactly one initial vertex, this number equals $|\mathbf{E}|$. Similarly $\mathrm{d}^-(\mathbf{u}) = |\mathbf{E}|$, since every directed edge has exactly one terminal vertex:

$$\sum_{\mathbf{u} \in \mathbf{V}} \deg^+(\mathbf{u}) = \sum_{\mathbf{u} \in \mathbf{V}} \deg^-(\mathbf{u}) = |\mathbf{E}|.$$

Lemma A.7 (**Handshaking Lemma**) In any directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$:

$$\sum_{\mathbf{u} \in \mathbf{V}} \deg(\mathbf{u}) = 2|\mathbf{E}|.$$

Corollary A.8 Any directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ has an even number of vertices which are of odd degree $\deg(\mathbf{u})$.

$$\underbrace{2|\mathbf{E}|}_{\text{even}} - \underbrace{\sum_{\mathbf{u}\in\mathbf{V}_2} \deg(\mathbf{u})}_{\text{even}} = \underbrace{\sum_{\mathbf{u}\in\mathbf{V}_1} \underbrace{\deg(\mathbf{u})}_{\text{odd}}}_{\textit{must be even}}$$